Linear integer programming
Branch and Bound methodology (B&B)

$$\max \ x_0 = \mathbf{c}^T\mathbf{x},$$
$$\mathbf{A}\ \mathbf{x} = \mathbf{b},$$
$$\mathbf{x} \geq \mathbf{0},\ \mathbf{x} \in Z^n$$

- **The base of the B&B methodology is the overview of solutions, determined as a tree.**
- **The finite dimension of a feasible solution is taken under consideration in the case of bounded ILP.**
- **Steps: -division**

  -branch

  - bounds as lower and upper values of the objective function.

Branch and Bound methodology - B&B

$$S_j = \{x | A^j x = b^j, x \geq 0 \text{ i } x \in Z^n\},$$

Transformation to the linear programming problem LP:

$$T_j = \{x | A^j x = b^j, x \geq 0\}$$
$$T_j \supseteq S_j$$

Branch and Bound methodology – B&B

***Partition.* The solution of LP problem in not integer.**

Let

$$x_{Bi} = [y_{i0}] + f_{i0}, \quad 0 < f_{i0} < 1.$$

Partition S_j is as follows:

$$S_j^* = \{S_j \cap \{x | x_{Bi} \leq [y_{i0}]\}, S_j \cap \{x | x_{Bi} \geq < y_{i0} >\}\},$$

Where:

<a> is a minimal integer value greater or equal value „a", and [a] is a greatest, integer value less or equal value „a".

Branch and Bound method – B&B

Let us take the assumption, that each variable x_j is upper bounded by u_j value .

Let

$$S_k = \{x | Ax = b, 0 \leq \alpha_j^k \leq x_j \leq \beta_j^k \leq u_j, \text{ integer } \quad j = 1,...,n\},$$
$$H_k = \{x | 0 \leq \alpha_j^k \leq x_j \leq \beta_j^k \leq u_j, x_j \text{ integer } \quad j = 1,...,n\}.$$

- The LP problem is a weakened ILP problem.
- The linear programming problem LP with the constraint for lower and upper value of variable will be solved with a dual simplex algorithm.

Methodology Branch and Bound

- Based on „branch and bound" technique
- Let us choose a variable to branch (which takes the real values and solve the linear programming problem LP
- The optimal objective function value of LP problem is an upper bound of optimal value of an objective function for ILP problem.
- The objective function value for integer solution of ILP problem is a lower bound of optimal objective function value for ILP problem.

ILP = LP + constraints for integer values of variables

Constraints for a variable range

- For variables, which values are not integer:

$$d_j \leq x_j \leq g_j$$
$$d_j \leq x_k \leq [x_k^0] \qquad\qquad [x_k^0] + 1 \leq x_k \leq g_k$$

Let take:

$$d_j = 0 \qquad g_j = M$$

M - Great enough integer value

Constraints for variables

- In the feasible set of solutions in LP problem following constraints are added:

$$\left[x_k^0\right] < x_k < \left[x_k^0\right] + 1$$

It leads to divide that set for two subsets.

Example of integer linear programming problem ILP

$$\max\ x_0 = 6x_1 + 5x_2$$

$$9x_1 + 7x_2 \le 63$$
$$x_1 + x_2 \le 8$$
$$3x_1 + 2x_2 \ge 6$$
$$x_1, x_2 \ge 0$$

Solution of LP

		x_5	x_4
x_0	43,5	0,5	0,5
x_3	13,5	0,5	-1,5
x_2	4,5	-0,5	4,5
X_1	3,5	0,5	-3,5

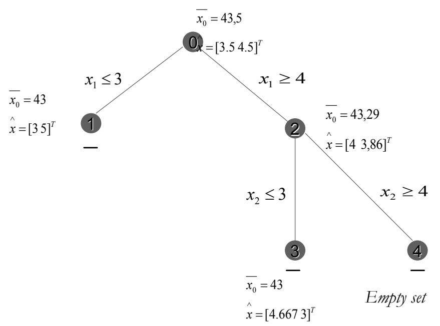
$$\hat{x} = \begin{bmatrix} 3,5 \\ 4,5 \end{bmatrix}, \quad \hat{x}_0 = 43,5$$

Solution of ILP

		x_6	x_4
x_0	**43**	1	5
x_3	**1**	-2	-7
x_2	**5**	-1	1
X_1	**3**	1	0
X_5	**13**	1	2

$$\hat{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \hat{x}_0 = 43$$

Solutions tree



Example of integer linear programming problem ILP

$$\max\ x_0 = -7x_1 - 3x_2 - 4x_3$$

$$x_1 + 2x_2 + 3x_3 - x_4 = 8$$
$$3x_1 + x_2 + x_3 - x_5 = 5$$
$$x_1, x_2, x_3 \ge 0$$

Solution of LP

		$-x_1$	$-x_3$	$-x_5$
x_0	-14.2	2.2	0.6	0.4
x_2	3.8	0.2	1.6	-0.6
x_1	0.4	-0.4	-0.2	0.2

Solution of ILP

		$-x_1$	$-x_3$	$-x_5$
x_0	-15	2	1	3
x_2	5	3	1	-1
x_4	2	5	-1	-2

The indirect review – the branch and bound methodology for binary variables

- The binary condition for a variable is not an important constraint, if the upper bound for variable x_j is known:

$$dla\ x_j \in Z\ i\ 0 \le x_j \le u_j$$
$$x_j \in S_j = \{s_{1j}, \ldots, s_{pj}\}$$

It is equivalent to the following set of constraints:

$$x_j = \sum_{k=1}^{p} s_{kj}\delta_{kj}$$

$$\sum_{k=1}^{p} \delta_{kj} = 1\ for\ each\quad j$$

$$\delta_{kj} = 0\ or\ 1, \quad k = 1,2,\ldots,p\ for\ each\ j$$

The indirect review – the branch and bound methodology for binary variables

Steps of algorithm:

- branch – try to choose one variable x_j and take

$$S_k^* = \{S_k \cap \{\mathbf{x}, x_j = 0\}, S_k \cap \{\mathbf{x}, x_j = 1\}\}$$

and

$$S_k^+ = \{j, j \in W_k\ i\ x_j = 1\}$$
$$S_k^- = \{j, j \in W_k\ i\ x_j = 0\}$$
$$F_k = \{j, j \notin W_k\}$$

The indirect review – the branch and bound methodology for binary variables

- bounds– each extreme point v_k defines the following problem:

$$\max\ z_k = \sum_{j\in F_k} c_j x_j + \sum_{j\in S_k^+} c_j,$$

$$\sum_{j\in F_k} a_{ij} x_j \le b_i - \sum_{j\in S_k^+} a_{ij} = s_i, \qquad i=1,\dots,m,$$

$$x_j = 0 \text{ or } 1, \quad j\in F_k.$$

Linear integer programming – cutting plane methodology

$$\max\ x_0 = c^T x,$$
$$x \in S = \{x \mid Ax = b, x \ge 0 \text{ i } x \in Z^n\}. \qquad (1)$$

Let us assume, that there exist $\bar{A}$ and $\bar{b}$

$$T = \{x \mid Ax = b, \bar{A}x = \bar{b}, x \ge 0\}$$

$S \subseteq T$ and a weakened problem to (1):

$$\max\ x_0 = c^T x, \quad x \in T$$

has integer optimal solution x_{opt}.

Then x_{opt} is an optimal solution of problem (1).

Cutting plane methodology

(2)
$$\max\ x_0 = c^T x,$$
$$x \in Q = \{x \mid Ax = b, x \ge 0\}.$$

Let us assume, that the formal representation of one row of problem (2) takes the following form:

$$x_{B_i} = y_{i0} - \sum_{j\in R_N} y_{ij} x_j, \quad i = 0,1,\dots,m,$$

Basic cutting plane

$$\sum_{j\in R_N} ([h]y_{ij} - [hy_{ij}])x_j \ge [h]y_{i0} - [hy_{i0}]$$

Cutting planes in integer programming metod

$$\sum_{j\in R_N} (y_{ij} - [y_{ij}])x_j \ge y_{i0} - [y_{i0}].$$

$$y_{ij} = [y_{ij}] + f_{ij}$$

$$\sum_{j\in R_N} f_{ij} x_j \ge f_{i0},$$

$$s = -f_{i0} + \sum_{j\in R_N} f_{ij} x_j, \qquad s \ge 0.$$

s has to be integer :

$$x_{Bi} = -(-f_{i0} + \sum_{j\in R_N} f_{ij} x_j) + ([y_{i0}] - \sum_{j\in R_N} [y_{ij}]x_j),$$

$[y_{i0}] - \sum_{j\in R_N} [y_{ij}]x_j$ is integer.

The optimal solution for integer linear programming problem ILP

The feasible solution of ILP problem is an optimal solution, when the following conditions are fulfilled:

(i) Primal feasibility $\quad y_{i0} \ge 0, \ i = 1,\dots,m;$

(ii) Integer value of $\quad y_{i0}$ for $i = 1,\dots,m;$

(iii) Dual feasibility $\quad y_{0j} \ge 0$ for all $j \in R_N$

Cutting plane algorithm for ILP

Step 1

Let us find a feasible solution, which fulfills two froim three necessary conditions. Go to Step 2.

Step 2 – Optimality test

If the third condition is fulfilled – then STOP. It is optimal integer solution.

In an opposite side – go to Step 3.

Step 3 – Cutting plane constraint and the elimination

Try to construct one cutting plane with a real value h and solve the linear programming problem with real variables.

Go to Step 2 .

Linear programming method for integer variables

$$\max_{x \in X} \; x_0 = 2\mathbf{x}_1 + 1x_2 \qquad X = \left\{ x: \begin{array}{c} x_1 + x_2 \le 5 \\ -x_1 + x_2 \le 0 \\ 6x_1 + 2x_2 \le 21 \end{array}, \begin{array}{c} x \ge 0 \\ x \in Z^n \end{array} \right\}$$

The solution for LP problem for $x \in R^n$ with a cutting plane for an additional variable s_1

		x_3	x_5
x_0	31/4	1/2	1/4
x_1	11/4	-1/2	1/4
x_2	9/4	3/2	-1/4
x_4	1/2	-2	1/2
s_1	-0,5	0	-0,5

The possible cutting planes:

$$\frac{1}{2}x_3 + \frac{1}{4}x_5 \ge \frac{3}{4}$$
$$\frac{1}{2}x_3 + \frac{1}{4}x_5 \ge \frac{3}{4}$$
$$\frac{1}{2}x_3 + \frac{3}{4}x_5 \ge \frac{1}{4}$$
$$\frac{1}{2}x_5 \ge \frac{1}{2}$$

Choosen cutting plane: $-\frac{1}{2}x_5 + s_1 = -\frac{1}{2}$

The iterations for cutting plane algorithm - with the help of dual simplex method.

An optimal table but not integer solution

		x_3	s_1
x_0	30/4	1/2	1/2
x_1	10/4	-1/2	1/2
x_2	10/4	3/2	-1/2
x_4	0	-2	1
x_5	1	0	-2

New cutting plane constraint s_2

		x_3	s_1
x_0	30/4	1/2	1/2
x_1	10/4	-1/2	1/2
x_2	10/4	3/2	-1/2
x_4	0	-2	1
x_5	1	0	-2
s_2	-1/2	-1/2	-1/2

		s_2	s_1
x_0	7	1	0
x_1	3	-1	1
x_2	1	3	-2
x_4	2	-4	3
x_5	1	0	-2
x_3	1	-2	1

Feasible, optimal and integer solution

$\mathbf{x=[x_1,x_2,x_3,x_4,x_5,s_1,s_2] = [3,1,1,2,1,0,0]}$

$\mathbf{x_0=7}$

How to choose a row to construct the cutting plane -heuristic rules

- ❑ The cutting plane constraint ought to cut the great surface of a feasible set.
- ❑ This part of a cutted feasible set doesn't contain the integer numbers.
  - ❑ The cutting plane has to be deep, when

$$f_{ij}\downarrow \; and \; f_{i0}\uparrow$$

  - ❑ f_{i0} has to be great enough but f_{ij} ought to be small enough for

$$j \in R_N$$

How to choose a row to construct the cutting plane -heuristic rules

(I) $$f_{r0} = \max_i \; f_{i0}$$

(II) $$\frac{f_{r0}}{\sum_{j \in R_N} f_{rj}} = \max_i \frac{f_{i0}}{\sum_{j \in R_N} f_{ij}}$$

(III) $$\frac{f_{r0}}{f_{rk}} = \max_i \frac{f_{i0}}{f_{ik}}$$

for defined $k \in R_N$

How to check the integer variables in ILP problem

The real value r will be determined as integer value, when

$$\min \{1 - f_r, f_r\} \le \varepsilon$$

If the integer value will be badly determined then it can lead to:

- ➢ unnecessary iterations,
- ➢ incorrect cutting planes
- ➢ a loss of an optimal solution.

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(ii) Integer value of y_{i0} for $i = 1,\dots, m;$

(iii) Dual feasibility, $y_{0j} \ge 0$ for all $j \in R_N$

Review of cutting plane methods

1. Integer form algoritm- the integer condition is not fulfilled

   for y_{i0} *i=1,...,m*

2. Integer dual algorithm – the primal feasibility condition is not fulfilled:

$$y_{i0} \geq 0 \ for \quad i = 1,...,m$$

3. Integer primal algorithm – the dual feasibility condition is not fulfilled:

$$y_{0j} \geq 0 \ for \quad \forall j \in \mathbf{R}_N$$