II The general linear programming problem LP

$$\max \; f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

under the constraints:

$$\mathbf{A}_1 \mathbf{x} \le \mathbf{b}_1$$
$$\mathbf{A}_2 \mathbf{x} \ge \mathbf{b}_2$$
$$\mathbf{x} \ge 0$$

dim ***x****=[n*1], dim* ***c****=[n*1]*

Matrices $\boldsymbol{A_1}, \boldsymbol{A_2}$:

dim $\boldsymbol{A_1}$ *=[m $_1$ * n], dim* $\boldsymbol{A_2}$ *=[m $_2$ * n]*

Vectors b_1, b_2 :

dim $\mathbf{b_1}$*=[m$_1$*1], dim* $\mathbf{b_2}$*=[m$_2$*1]*

The general linear programming problem LP

Two phase simplex method

I phase - it is necessary to find a first basic, admissible solution by solving the auxiliary problem with an auxiliary objective finction

II phase - maximization of an objective function x_0 with the simplex method.

Step1. ***(start).*** **Let us find first basic, admissible solution.**

The permissibility condition: if $y_{i0} \ge 0 \; for \; i = 1,\ldots,m$

Yes – go to Step 2, No – STOP.

I phase - it is necessary to find a first basic, admissible solution by solving the auxiliary problem with an auxiliary objective function

- The initial non - admissible simplex table can be transformed to the permissible form with the help of the simplex method.
- Aim – to obtain non negative values of auxiliary variables.
- It is necessary to determine a minimal value:

$$s \Rightarrow \min_i \{ y_{i0}, \; y_{i0} < 0 \; dla \; i = 1,2,\ldots,m \}$$

- The variable s will be the auxiliary objective function, which will be maximized.
- I phase is realised till the permissible condition will be fulfilledi:

$$y_{i0} \ge 0 \; dla \; i = 1,2,\ldots,m$$

- The end of I phase

Example III for linear programming problem two phase simplex method

$$\max_{x \in X} \; x_0 = 1x_1 + 6x_2$$

$$X = \left\{ x : \begin{array}{rrrcl} +2x_1 & + & 1x_2 & \ge & 2 \\ -1x_1 & + & 1x_2 & \le & 3 \\ +1x_1 & + & 1x_2 & \le & 6 \end{array}, x \ge 0 \right\}$$

I phase

		$-x_1$	$-x_2$
x_0	0	-1	-6
x_3	-2	-2	-1
x_4	3	-1	1
x_5	6	1	1

The table is not admissible

II phase part.1

		$-x_5$	$-x_2$
x_0	6	1	-5
x_3	10	2	1
x_4	9	1	2
x_1	6	1	1

I basic admissible solution

$$x_{B_1} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad x_0^{B_1} = 6$$

II phase part.2

		$-x_5$	$-x_4$
x_0	28,5	3,5	2,5
X_3	5,5	1,5	-0,5
X_2	4,5	0,5	0,5
X_1	1,5	0,5	-0,5

II basic admissible solution – optimal solution

$$x_{B_1} = \begin{bmatrix} 1,5 \\ 4,5 \end{bmatrix}, \quad x_0^{B_2} = 28,5$$

Optimal solution: $\hat{x} = \begin{bmatrix} \hat{x}^1 & \hat{x}^2 & \hat{x}^3 & \hat{x}^4 & \hat{x}^5 \end{bmatrix}^T == [1,5 \quad 4,5 \quad 5,5 \quad 0 \quad 0]^T$ $\quad \hat{x}_0 = 28,5$

III The linear programming problem LP for real variables

$$\max \; f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

under the constraints:

$$\mathbf{A}_1 \mathbf{x} \le \mathbf{b}_1$$
$$\mathbf{A}_2 \mathbf{x} \ge \mathbf{b}_2$$
$$\mathbf{x}_1 \in R, x_2 \ge 0$$

It is necessary to use the additional variable in the form: $\mathbf{x}_i = x_i^{'} - x_i^{''}$

$$\min_{x \in X} \; x_0 = \mathbf{x}_1 + x_2 \qquad x_1 := \mathbf{x}_1 - x_3 \qquad \max_{x \in X} \; x_0 = (\mathbf{x}_1 - x_3) + x_2$$

$$X = \left\{ x : \begin{array}{c} x_1 + x_2 \le 4 \\ -x_1 + x_2 \le 2 \\ x_1 \in R, x_2 \ge 0 \end{array} \right\} \Longrightarrow X = \left\{ x : \begin{array}{c} x_1 + x_2 - x_3 \le 4 \\ -x_1 + x_2 + x_3 \le 2 \\ x \ge 0 \end{array} \right\}$$

The basic, optimal solution: $\hat{x} = [x_1, x_2, x_3]^T = [0, 0, 2]^T \Longrightarrow \hat{x} = [x_1, x_2]^T = [-2, 0]^T$

The optimal, objective function: $\hat{x}_0 = -2$

The special case – the set of admissible solutions is empty

$$X = \phi$$

In the first phase of simplex method the algorithm could not construct the first, basic, admissible solution – because there is no admissible solution in this case of the linear programming problem.

Example: $\max_{x \in X} \; x_0 = \frac{1}{2}\mathbf{x}_1 - x_2 - x_3$

$$X = \left\{ x : \begin{array}{c} -\frac{1}{2}x_1 + 2x_2 + x_3 \le 2 \\ -\frac{1}{2}x_1 + 2x_2 - x_3 \ge 3 \\ x_2 - x_3 \le 2 \end{array}, x \ge 0 \right\}$$

		$-x_1$	$-x_2$	$-x_3$
x_0	0	-1/2	1	1
x_4	2	-1/2	2	1
x_5	-3	1/2	-2	1
x_6	2	0	1	-1

		$-x_1$	$-x_4$	$-x_3$
x_0	x	x	x	x
x_2	x	x	x	x
x_5	-1	0	1	2
x_6	x	x	x	x

From the assumption the variables are >=0 but from (1) it leads to : $x_5 < 0$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

$$x_5 = -x_4 - 2x_3 - 1 \qquad (1)$$

IV Linear programming problem LP

$$\min \quad x_0 = \mathbf{c}^T \mathbf{x}$$

Under the constraints in the form:

$$\mathbf{A\,x} \geq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

dim $\mathbf{x}$*=[n*1], dim* $\mathbf{c}$*=[n*1]*

dim A=[m x n]

dim $\mathbf{b}$ *=[m*1]*

Canonical form of IV LP problem

$$\min \ x_0 = c^{\mathbf{T}} x,$$
$$\mathbf{A} x \geq b,$$
$$x \geq \mathbf{0},$$

$$\max \ \text{-}\, x_0 = -c^{\mathbf{T}} x,$$
$$-\mathbf{A}\ x \ + I_s x_{\mathbf{s}} = -b,$$
$$x\ ,\ x_{\mathbf{s}} \geq \mathbf{0},$$

Optimal solution of IV LP problem with dual simplex method

Theorem:

The basic admissible solution of the set of equtions Ax=b is an optimal solution of IV LP problem, if the following conditions will be fulfilled:

(i) Dual admissible condition:

$$y_{oj} \geq 0 \quad dla \quad j \in R_N$$

(ii) Dual optimal condition:

$$y_{i0} \geq 0 \quad dla \quad i \in \{1,\ldots,m\}$$

Dual simplex algorithm for the over-equal linear constraints

Step1. ***(start).*** **Let us find first dual basic, admissible solution.**

The admissibility condition: if $y_{oj} \geq 0 \ dla \ j \in R_N$

Yes – go to Step 2, No – STOP.

Step2. ***(optimality condition).*** **If** $y_{i0} \geq 0$ **for each** $i = 1,\ldots,m$ **?**

- **Yes - an actual solution is optimal.**
- **No - go to Step 3.**

Step3. **(the variable, which will remove from the base). Let us choose such variable** x_{B_r} **for which** $y_{r0} < 0.$

The rule to select the variable x_{B_r} **is as follows:**

$$y_{r0} = \min_{j \in R_N} \{ y_{io},\ y_{i0} < 0,\ i = 1,\ldots,m \}$$

Go to Step 4.

Dual simplex algorithm

Step 4. (the variable, which will enter to the base). Let us choose such variable x_k

For which, the following relation is fulfilled:

$$\frac{y_{0k}}{y_{rk}} = \max \left(\frac{y_{0j}}{y_{rj}}, y_{rj} < 0 \right).$$

If some variables exist with this condition, let us choose only one arbitrary.

Go to Step 5.

Step 5. (Gauss method).

Calculate the dual simplex iteration with Gauss elimination method entering the variable x_k **to the base and moving the variable** x_{B_r} **out from the base.**

Let us substitute for a new basic, admissible solution.

Go to Step 2.

Example for the dual simplex method

$$\min_{x \in X} \ x_0 = 1\mathbf{x}_1 + 1x_2$$

$$X = \left\{ x : \begin{array}{lllll} +1x_1 & + & 2x_2 & \geq & 8 \\ 2x_1 & + & 1x_2 & \geq & 6, x \geq 0 \\ +1x_1 & + & 1x_2 & \geq & 5 \end{array} \right\}$$

Initial table

		$-x_1$	$-x_2$
$-x_0$	0	1	1
X_3	-8	-1	-2
X_4	-6	-2	-1
X_5	-5	-1	-1

Dual admissible table

$$y_{oj} \geq 0 \ dla \ j \in R_N$$

Intermediate table

		$-x_1$	$-x_3$
$-x_0$	-4	½	½
X_2	4	½	-1/2
x_4	-2	-3/2	-1/2
X_5	-1	-1/2	-1/2

Not yet optimal table

$$y_{20} < 0$$
$$y_{30} < 0$$

Intermediate table

		$-x_4$	$-x_3$
$-X_0$	-14/3	1/3	1/3
X_2	10/3	1/3	-2/3
X_1	4/3	-2/3	1/3
X_5	-1/3	-1/3	-1/3

Not yet optimal

$$y_{30} < 0$$

Example for the dual simplex method

I optimal table

II optimal table

$y_{i0} \geq 0 \quad dla \forall i = 1,..., m$

		$-x_5$	$-x_3$
$-X_0$	-5	1	0
X_2	3	1	-1
X_1	2	-2	1
X_4	1	-3	1

		$-x_5$	$-x_4$
$-x_0$	-5	1	0
X_2	4	-2	1
x_1	1	1	-1
X_3	1	-3	1

I extreme point: $\hat{x}^1 = [x_1^1, x_2^1, x_3^1, x_4^1, x_5^1] = [2, 3, 0, 1, 0]$

II extreme point $\hat{x}^2 = [x_1^2, x_2^2, x_3^2, x_4^2, x_5^2] = [1, 4, 1, 0, 0]$ $\quad \hat{x}_0 = 5$

The set of optimal solutions: $\hat{x} = \{x : x = (1+\lambda), (4-\lambda), \quad \lambda \in [0,1]\}.$

Dual theory for linear programming problem LP

LP problem

$$\max \; x_0 = \mathbf{c}^T \mathbf{x}$$
$$\mathbf{A}\,\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

Dual LP problem

$$\min \; v_0 = v^T b$$
$$A^T \, v \geq c$$
$$v \geq 0$$

Theorem 7.1 :

If a vector x is an admissible solution for LP problem and a vector v is an admissible solution for dual LP problem, then the optimal value of dual LP problem fulfills the following relation:

$$v^T b \geq c^T x$$

Dual theory for linear programming problem LP

Theorem 7.2

If one of the two problems: LP problem or dual LP problem has its optimal solution,

there the second problem has also its optimal solution $\hat{x}$ *i* $\hat{v}$ **and two problems have the same objective function value.**

$$c^T \hat{x} = b^T \hat{v}$$

Theorem 7.3

Let (x_r, x_s) i (v_r, v_s) be the admissible solutions for LP problem and dual LP problem in the canonical form of equations.

Then the complementary condition for variables in LP problem and dual LP problem will take the form as below:

$$\hat{x}^T \hat{v} = 0 \quad \Longrightarrow \quad x_r^T v_s + x_s^T v_r = 0$$

Theorem 7.4

If dual LP problem is unlimited, then the LP problem is inconsistent.

Example I LP problem

$$\min_{v \in V} v_0 = 5v_1 + 0v_2 + 21v_3$$
$$V = \left\{ v : \begin{array}{c} v_1 - v_2 + 6v_3 \geq 2 \\ v_1 + v_2 + 2v_3 \geq 1 \\ v_1, v_2, v_3 \geq 0 \end{array} \right\}$$

Dual LP problem

$$\max_{x \in X} x_0 = 2x_1 + 1x_2$$
$$X = \left\{ x : \begin{array}{c} x_1 + x_2 \leq 5 \\ -x_1 + x_2 \leq 0 \\ 6x_1 + 2x_2 \leq 21 \end{array} , x \geq 0 \right\}$$

Solution vector:

$$v = [v_r, v_s] = [v_1, v_2, v_3 / v_4, v_5] \qquad x = [x_r, x_s] = [x_1, x_2 / x_3, x_4, x_5]$$

$$x^T v = 0 \Rightarrow \quad [x_r, x_s]^T [v_s, v_r] = 0$$

Example II **Cutting stock problem**

$$\min v_0 = 0.3v_1 + 0.6v_2 + 0.2v_3$$
$$7v_1 + 3\,v_2 + 0v_3 \geq 2100$$
$$0v_1 + 1v_2 + 2v_3 \geq 1200$$
$$v_1, v_2, v_3 \geq 0$$

$$\max_{x \in X} x_0 = 2100\mathbf{x}_1 + 1200x_2$$
$$X = \left\{ x : \begin{array}{c} 7x_1 + 0x_2 \leq 0.3 \\ 3x_1 + 1x_2 \leq 0.6 \\ 0x_1 + 2x_2 \leq 0.2 \end{array} , x \geq 0 \right\}$$

Dual theory for linear programming problem LP

I. Dual LP problem solution with simplex method

Dual LP problem: $\max_{x \in X} x_0 = 2\mathbf{x}_1 + 1x_2 \quad X = \left\{ x : \begin{array}{c} 1x_1 + 1x_2 \leq 5 \\ -x_1 + x_2 \leq 0 \\ 6x_1 + 2x_2 \leq 21 \end{array} , x \geq 0 \right\}$

		x_1	x_2
x_0	0	-2	-1
x_3	5	1	1
x_4	0	-1	1
x_5	21	6	2

		x_5	x_2
x_0	7	1/3	-1/3
x_3	3/2	-1/6	2/3
x_4	7/2	1/6	4/3
x_1	7/2	1/6	1/3

		x_5	x_3
x_0	31/4	1/4	1/2
x_2	9/4	-1/4	3/2
x_4	1/2	1/2	-2
x_1	11/4	1/4	-1/2

Optimal solution:

a) Dual LP problem $\quad \hat{x} = [x_1, x_2, \uparrow x_3, x_4, x_5] \quad \hat{x} = \left[\frac{11}{4}, \frac{9}{4}, \uparrow 0, \frac{1}{2}, 0\right]$

b) LP problem $\quad \hat{v} = [v_4, v_5, \uparrow v_1, v_2, v_3] \quad \hat{v} = \left[0, 0, \uparrow \frac{1}{2}, 0, \frac{1}{4}\right]$

Optimal value of the objective function $\quad \hat{x}_0 = \hat{v}_0 = \frac{31}{4}$