Linear programming - summary

- **Each basic admissible solution is an extreme point of the admissible solutions set X.**
- **There exists an extreme point of the set X, where objective function of a linear programming problem reaches its maximal value.**

Special cases

II. The linear programming problem – the unlimited problem

<u>Theorem 6</u>

If the LP problem is unlimited, there

the basic, admissible solutions x_B **exist and a vector** y_K **, such that:**

$$y_{0k} < 0 \; i \; y_{ik} \le 0 \; dla \; i = 1,\dots,m \quad .$$

Then the solution set is empty.

Example:

$$\min_{x \in X} \; x_0 = -\mathbf{x}_1 - x_2$$

$$X = \left\{ x : \begin{matrix} -x_1 & - & x_2 & \le & 2 \\ -x_1 & + & x_2 & \le & 1 \end{matrix}, x \ge 0 \right\}$$

Special cases
III. The linear programming problem – an infinite number of optimal solutions belonging to the finite set.

- In this case the simplex table corresponding to the basic, optimal solution, such that:

$$y_{i0} \ge 0 \; for \; i = 1,\dots,m \; and \qquad y_{0j} \ge 0 \; for \; j = 1,\dots,n$$

there exist the pair of indexes $(i_0, j_0),\, i_0 \in \{1,\dots,m\},\, j_0 \in \{1,\dots,n\}$

for which: $$y_{0j_0} = 0,\; y_{i_0 0} > 0 \; i \; y_{i_0 j_0} > 0$$

- In the space with a dimension equal *n* the optimal solution is a convex combination of x^i

extreme points: $$\hat{x} = \sum_{i=1}^{p} \lambda_i \hat{x}^i \quad where \quad \sum_{i=1}^{p} \lambda_i = 1, \quad \lambda_i \in [0,1], \; i = 1,\dots,p$$

Special cases
III. Example for the linear programming problem – an infinite number of optimal solutions belonging to the finite set.

$$\max_{x \in X} \; x_0 = 4\mathbf{x}_1 + 2x_2 \qquad X = \left\{ x : \begin{matrix} -x_1 & + & x_2 & \le & 4 \\ 2x_1 & + & x_2 & \le & 6 \end{matrix}, x \ge 0 \right\}$$

		x_1	x_2
x_0	0	-4	-2
x_3	4	-1	1
x_4	6	2	1

		x_4	x_2
x_0	12	2	0
x_3	7	0.5	1.5
x_1	3	0.5	0.5

		x_4	x_3
x_0	12	2	0
x_2	4.66	0.33	0.66
x_1	0.66	0.33	-0,33

- 1 basic, optimal solution in : R^2 $\quad \hat{x}^1 = [3,0]^T$
- 2 basic optimal solution in : R^2 $\quad \hat{x}^2 = [\frac{2}{3}, \frac{14}{3}]^T$

The two basic, optimal solutions correspond to two vertexes of the set of optimal solutions.

The set of optimal solutions:

$$\hat{X} = \left\{ \hat{x} : \hat{x} = \lambda \hat{x}^1 + (1-\lambda)\hat{x}^2, \lambda \in [0,1] \right\} = \left\{ \hat{x} : \hat{x} = (3\lambda + \frac{2}{3}(1-\lambda)), (0 + \frac{14}{3}(1-\lambda)), \; \lambda \in [0,1] \right\}$$

$$= \left\{ \hat{x} : \hat{x} = (\frac{2}{3} + \frac{7}{3}\lambda), (\frac{14}{3} - \frac{14}{3}\lambda) \right\}.$$

Special cases
IV. The linear programming problem – an infinite number of optimal solutions belonging to the finite set.

This case will be, when in a simplex table an optimal, basic solution will be calculated,

such that: : $y_{i0} \ge 0 \; for \; i = 1,\dots,m \;\; and \;\; y_{0j} \ge 0 \; for \; j = 1,\dots,n$

And there exists index j_0 ⟹ $y_{0j_0} = 0$ and

for all indexes *„i" two cases can apear:*

1. $y_{i0} = 0$ (degeneration) or
2. $y_{ij_0} \le 0$

IV. The linear programming problem – an infinite number of optimal solutions belonging to the infinite set.

- The optimal solution of LP problem takes the parametric form:
  - for n=2 the parametric equation for a straight line,
  - for n=3 the parametric equation for a surface etc.

Example:

$$\max_{x \in X} \; x_0 = -2\mathbf{x}_1 + 4x_2 \qquad X = \left\{ x : \begin{matrix} -2x_1 & + & x_2 & \le & 1 \\ -1x_1 & + & 2x_2 & \le & 4 \end{matrix}, x \ge 0 \right\}$$

		x_1	x_2
x_0	0	2	-4
x_3	1	-2	1
x_4	4	-1	2

		x_4	x_3
x_0	4	-6	4
x_2	1	-2	1
x_4	2	3	-2

		x_4	x_3
x_0	8	2	0
x_2	2.33	0.66	-0,33
x_1	0.66	0.33	-0,66

The basic, optimal solution: $\hat{x}^1 = [\frac{2}{3}, \frac{7}{3}]^T$

The set of optimal solutions is related to the ray: $\hat{X} = \left\{ \hat{x} \in R^2 : \hat{x} = \hat{x}^1 + [\frac{2}{3}, \frac{1}{3}]^T t, \quad t \ge 0 \right\}$

V. The linear programming problem – the set of admissible solutions is empty – there is no solution.

$$X = \phi$$

Example:

$$\max_{x \in X} \; x_0 = \mathbf{x}_1 + 4x_2$$

$$X = \left\{ x : \begin{matrix} x_1 & + & x_2 & \geq & 4 \\ x_1 & + & x_2 & \leq & 2 \end{matrix}, x \geq 0 \right\}$$

		x_1	x_2
x_0	0	-1	-4
x_3	-4	-1	-1
x_4	2	1	1

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -4 \\ +2 \end{bmatrix}$$

The admissible condition is not fulfilled. It is impossible to recognize the case.

Step1. **(*start*). Let us find first basic, admissible solution.**

The admissible condition: if $y_{i0} \geq 0 \; dla \; i = 1,\ldots,m$

Yes – go to Step 2, No – STOP.