OPTIMIZATION THEORY AND ADVANCED COMPUTING METHODS

The linear programming problem - Part II

Faculty of Electronics
Embedded Robotics
M.Sc. level

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Extreme point for the linear programming problem LP

Theorem 1

The admissible solution x for the linear programming problem LP is a vertex of a set of admissible solutions *X* if and only if

the basic, admissible solution is related to this vertex, as follows:

$$x = [x_B, 0]^T$$

1. Let us assume, that the vector x is a basic admissible solution of the LP problem. It is necessary to show, that x is an extreme point of a set X.
2. Let us assume, that the vector x is an extreme point of an admissible set of solution for the linear programming problem LP. It is necessary to show, that x is a basic, admissible solution of LP.

Extreme points for the linear programming problem PL cd.

Theorem 2

1. The objective function of the linear programming problem PL achieves its maximal value in extreme point of an admissible set of solutions X.
2. If the objective function of the linear programming problem PL achieves its maximal values in more then one extreme point, it has the same value for each convex combination of these points.

For p basic admissible optimal solutions: $\hat{x}^1, \hat{x}^2 \ldots\ldots \hat{x}^p$

The optimal set of solutions takes the following form:

$$\hat{X} = \sum_{i=1}^{p} \lambda_i \hat{x}^i, \;\; \lambda_i \geq 0, i = 1,\ldots,p \;\; \sum_{i=1}^{p} \lambda_i = 1$$

Notations:

$$y_0 \equiv \begin{bmatrix} c_B B^{-1} b \\ B^{-1} b \end{bmatrix} \equiv \begin{bmatrix} y_{00} \\ y_{10} \\ \vdots \\ y_{m0} \end{bmatrix} \quad \text{and} \quad y_j \equiv \begin{bmatrix} c_B B^{-1} a_j - c_j \\ B^{-1} a_j \end{bmatrix} \equiv \begin{bmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{mj} \end{bmatrix}$$

where $a_j, j \in R_N$ denote the columns of matrix N.

The objective function x_0

$$x_0 = y_{00} - \sum_{j \in R_N} y_{0j} x_j$$

m constraints:

$$x_{Bi} = y_{i0} - \sum_{j \in R_N} y_{ij} x_j, \quad for\ i = 1,\ldots,m$$

Initial form of simplex table:

		$-x_N$
x_0	$c_B B^{-1} b$	$c_B B^{-1} N - c_N$
x_B	$B^{-1} b$	$B^{-1} N$

$\Rightarrow$ $c_B = 0$

		$-x_N$
x_0	0	$-c_N$
x_B	b	N

where $c_B = 0$ then a first basic, admissible solution conforms for the extreme point x

$$x = [0,\ldots,0]^T.$$

An increase of a basic, admissible solution

$$x_0 = y_{00} - \sum_{j \in R_N} y_{0j} x_j$$

$$always\ \ x_j \geq 0,$$

$$then\ when\ x_k \uparrow \quad \text{simultaneously} \quad x_0 \uparrow when\ y_{0k} \leq 0$$

$$x_{Bi} = y_{i0} - \sum_{j \in R_N} y_{ij} x_j, \quad for\ i = 1,\ldots,m$$

Improvement of a basic, admissible solution – Gauss method .

$$\theta_{rk} = \min_{i=1,\dots,m} (y_{i0} / y_{ik}, y_{ik} > 0).$$

$$x_k = \frac{y_{r0}}{y_{rk}} - \sum_{j \in R_N - \{k\}} \frac{y_{rj}}{y_{rk}} x_j - \frac{1}{y_{rk}} x_{Br}$$

$$x_{B_i} = \left(y_{i0} - \frac{y_{ik} y_{r0}}{y_{rk}} \right) - \sum_{j \in R_N - \{k\}} \left(y_{ij} - \frac{y_{ik} y_{rj}}{y_{rk}} \right) x_j + \frac{y_{ik}}{y_{rk}} x_{Br}$$

If we can find a non-degenerated basic admissible solution such, that:

$y_{0j} < 0$ *for some* $j = k, k \in R_N$ *oraz* $y_{ik} > 0$

For at least one index „i" – then

We can obtain a better basic, admissible solution by changing one column from basic matrix B for one column from non-basic matrix N.

Improved simplex table – which coresponds to the next basic admissible solution with a better function value.

		...	$-x_j$	...	$-x_{Br}$	...
x_0	$y_{00} - (y_{0k} y_{r0} / y_{rk})$	...	$y_{0j} - (y_{0k} y_{rj} / y_{rk})$	...	$-y_{0k} / y_{rk}$	
		...		...		...
x_{Bi}	$y_{i0} - (y_{ik} y_{r0} / y_{rk})$	...	$y_{ij} - (y_{ik} y_{rj} / y_{rk})$		$-y_{ik} / y_{rk}$	
		...		...		...
x_k	y_{r0} / y_{rk}	...	y_{rj} / y_{rk}	...	$1 / y_{rk}$	...

The admissibility and optimality conditions for PL problem

The optimality conditions $y_{0j} \geq 0 \quad for \quad \forall j \in R_N$

The admissibility conditions

$y_{io} \geq 0$
for $i \in \{1, \dots, m\}$

		...	$-x_j$	...	$-x_k$	...
x_0	y_{00}	...	y_{0j}	...	y_{0k}	...
		...		...		...
x_{Bi}	y_{iO}	...	y_{ij}	...	y_{ik}	...
		...		...		...
x_{Br}	y_{r0}	...	y_{rj}	...	y_{rk}	...
		...		...		...
x_{Bm}	y_{m0}	...	y_{mj}	...	y_{mk}	...

The optimal solution of a linear programming problem solved by simplex method

Theorem 3

The basic, admissible solution of a set of equations *Ax=b*

is optimal for linear programming problem if the following conditions are fulfilled:

(i) The admissibility condition:

$$y_{io} \geq 0 \quad dla \quad i \in \{1, \dots, m\}$$

(ii) The optimality condition:

$$y_{0j} \geq 0 \quad dla \quad \forall j \in R_N$$

Simplex algorithm for the less-equal linear constraints part I

Step1. ***(start).*** **Let us find first basic, admissible solution.**

The admissibility condition: if $y_{i0} \geq 0 \; dla \; i = 1, \dots, m$

Yes – go to Step 2, No – STOP.

Step2. ***(optimality condition).*** **If** $y_{0j} \geq 0$ **for each** $j \in R_N$ **?**

- **Yes - an actual solution is optimal.**
- **No - go to Step 3.**

Step3. ***(The variable, which will enter to the base).*** **Let us choose such variable** $x_k, k \in R_N$ **for which** $y_{0k} < 0.$

The rule to select the variable x_k **is as follows:**

$$y_{0k} = \min_{j \in R_N} \{ y_{0j}, y_{0j} \leq 0 \}$$

Go to Step 4.

Simplex algorithm part I

Step 4. (the variable, which will remove from the base). **Let us choose such variable** x_{Br}**,**

For which, the following relation is fulfilled:

$$\theta_{rk} = \min_{i=1,\dots,m} (y_{i0} / y_{ik}, y_{ik} > 0).$$

If some variables exist with this condition, let us choose only one arbitrary.

Go to Step 5.

Step 5. (Gauss method). **Calculate** x_k **and** $x_{Bi}, i \neq R_N,$ **taking under consideration variables** $x_j, j \in R_N - \{k\}$ **and variable** x_{B_r} **according to the formulas on the slide 8**

Let us substitute $x_j = 0, j \in R_N - \{k\} \; \mathrm{i} \; x_{Br} = 0$ **for a new basic, admissible solution.**

Go to Step 2.

Example I of linear programming problem LP

Assumptions:

1. A set X exists $X \neq \phi$

2. A simplex algorithm starts from basic, admissible solution

I. The linear programming problem has one solution

$$\max_{x \in X} \; x_0 = 2\mathbf{x}_1 + 1x_2 \qquad X = \left\{ x: \begin{array}{c} x_1 + x_2 \le 5 \\ -x_1 + x_2 \le 0 \\ 6x_1 + 2x_2 \le 21 \end{array}, x \ge 0 \right\}$$

		x_1	x_2
x_0	0	-2	-1
x_3	5	1	1
x_4	0	-1	1
x_5	21	6	2

		x_5	x_2
x_0	7	1/3	-1/3
x_3	3/2	-1/6	2/3
x_4	7/2	1/6	4/3
x_1	7/2	1/6	1/3

		x_5	x_3
x_0	31/4	¼	1/2
x_2	9/4	-1/4	3/2
x_4	1/2	1/2	-2
x_1	11/4	1/4	-1/2

The basic, optimal solution: $\hat{x} = [x_1, x_2, x_3, x_4 . x_5]^T = [\frac{11}{4}, \frac{9}{4}, 0, \frac{1}{4}, 0]^T$

The optimal value of the objective function: $x_0 = \frac{31}{4}$

Optimal solutions for linear programming problem

__Theorem__

If linear programming problem LP problem has optimal solution and all basic solutions have non-degenerated form

then by simplex method we can achieve optimal solution in at most

$$\binom{n}{m}$$

iterations.

__Theorem 5.1 – Extreme solution for a mathematical programming problem__

When $f(x): R^n \to R$

Is concave function and X is a bounded, closed and convex set, there exists extreme point (or many extreme points) of a set X, where the function f(x) achieves the global minimum value on this set.

__Proof:__ Denial of the thesis $f(\hat{x}) < f(v^i)$

$\hat{x}$ - any global minimum

v - any extreme point of a set ***X.***

*A set **X is a bounded, closed and convex set,** a function f(x) **is continuous*** – there for the point $\hat{x}$ and the extreme points $v^i, i = 1,...,k$ exist also.

Each point $x \in X$, which is not an extreme vertex, can be determined as a convex combination of points v^i

$$\hat{x} = \sum_{i=1}^{k} \mu_i v^i, \quad \mu_i \ge 0, \quad i = 1,...,k; \quad \sum_{i=1}^{k} \mu_i = 1$$

The linear function f(x) is convex, so for any point $\hat{x} \in X$ **the following relation is fulfilled:**

$$f(\hat{x}) = f(\sum_{i=1}^{k} \mu_i v^i) \ge \sum_{i=1}^{k} \mu_i f(v^i) > \sum_{i=1}^{p} \lambda_i f(\hat{x}) = f(\hat{x})$$

which completes the proof.

__Colorary 5.1__

The set of admissible solutions $X \subset R^n$ for the linear programming problem is a convex set (a convex polyhedron).

__Colorary 5.2__

The linear function f(x) defined on a polyhedron (the convex set of admissible solutions) achieves it's bounds on the extreme points of a polyhedron.