OPTIMIZATION THEORY AND ADVANCED COMPUTING METHODS

The linear programming problem - Part I

Faculty of Electronics
Embedded Robotics
M.Sc. level

Ph.D. Eng. Ewa Szlachcic
Department of Automation, Mechatronics and Control Systems
Wrocław University of Science and Technology

The linear programming problem with minority restrictions LP

$$\max_{x \in X} f(\mathbf{x}) = \mathbf{c}^T \mathbf{x}$$

under the constraints

$$\mathbf{A}\mathbf{x} \le \mathbf{b}$$

$$\mathbf{x} \ge 0$$

dim **x**=n, dim **c**=n, dim **A** =[m x n], dim $\mathbf{b_1}$=m_1,

Canonical form of PL

$$\max_{x \in X'} x_0 = \mathbf{c'}^T \mathbf{x'}$$

$$X' = \{x' : A'x' = b, x' \ge 0\}$$

The equivalence of mathematical programming problems

I Equality constraints

$$g_i(x) = 0 \Rightarrow \begin{cases} g_i(x) \le 0 \\ g_i(x) \ge 0 \Rightarrow -g_i(x) \end{cases}$$

II Nonequality constraints – introduce a complementary variable

$$g_i(x) \pm x_{n+j} = 0$$

$$x_{n+j} \ge 0$$

III A variable with real values – as a difference of two non-negative variables

$$x_i \in R$$

$$x_i = x_i^+ - x_i^-$$

where: $x_i^+ \ge 0, x_i^- \ge 0$

Example

$$\max_{x \in X} x_0 = 2\mathbf{x}_1 + 1x_2 \qquad X = \left\{ x : \begin{array}{l} x_1 + x_2 \le 5 \\ -x_1 + x_2 \le 0 \\ 6x_1 + 2x_2 \le 21 \end{array}, x \ge 0 \right\}$$

- •Number of variables n=2,
- •Number of constraints m=3.

Canonical form of PL problem – introducing complementary variables for the set of m linear equations.

$$\max_{x \in X} x_0 = 2\mathbf{x}_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5$$

$$X = \left\{ x : \begin{array}{rrrrrcr} x_1 + & x_2 + & x_3 + & 0 + & 0 & = & 5 \\ -x_1 + & x_2 + & 0 + & x_4 + & 0 & = & 0 \\ 6x_1 + & 2x_2 + & 0 + & 0 + & x_5 & = & 21 \end{array} \right\}$$

Columns, which belongs to unit basic matrix B.

Canonical form of linear programming problem PL

$$\max_{x \in X} x_0 = 2\mathbf{x}_1 + 1x_2 \qquad X = \left\{ x : \begin{array}{l} x_1 + x_2 \le 5 \\ -x_1 + x_2 \le 0 \\ 6x_1 + 2x_2 \le 21 \end{array}, x \ge 0 \right\}$$

- n=5,
- m=3.

$$\max_{x \in X} x_0 = \mathbf{c}^T \mathbf{x} \qquad x^T = [x_1, x_2, x_3, x_4, x_5]^T \qquad c^T = [c_1, c_2, c_3, c_4, c_5]^T = [2, 1, 0, 0, 0]$$

$$X = \{x : Ax = b, x \ge 0\}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 6 & 2 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 0 \\ 21 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \ge 0$$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

Variables, which belong to unit basic matrix B: x_3, x_4, x_5

Basic definitions

For linear equations Ax=b, where rank of the matrix r[A_r]=r[A] three cases can exist:

1. **r[A] = m = n one solution of Ax=b exists.**
2. **r[A] = n < m there one solution exists, but there are (m - n) equations, which are redundant.**
3. **r[A] = m < n there are infinite number of solutions of a set Ax=b.**

Definition 4.1 basic matrix B

The basic matrix B of a set of equations Ax = b and r[A] = m, n>m is defined as nonsingular quadratic matrix with dim(m*m), consiting of linearly independant column a^j of a matrix A.

Definition 4.2 basic solution x_b

The basic solution of a set of equations Ax=b r[A]=m, n>m is defined as a vector

$x_b = B^{-1}b$, constructed with variables, corresponding to columns a^j of basic matrix B.

Elements of a basic vector x_b are named the basic variables.

Basic solutions

$A=[B,N]$ B- nonsingular basic matrix; N- non-basic matrix $\quad \det B \neq 0, \dim B = m$

$x=[x_B, x_N]$ x_B - The vector of basic variables for matrix *B*; x_N - The vector of non-basic variables for matrix N

Definition 4.3

Basic solution is basic admissible solution of linear programming problem PL, when the vector x_B is as follows: $x_B \geq 0$

Then the basic admissible solution consists of not greater number of „m" non-negative x_j.

$$x=[x_B,0] \quad x_B = B^{-1}b \quad x_N = 0$$

Definition 4.4

Non-degenerate basic admissible solution of problem PL is named as an admissible solution, where all variables x_i are greater zero.

First basic solution:

$$x_B = B^{-1}b - B^{-1}N\,x_N$$

$$x_0 = c_B\,B^{-1}b - (c_B\,B^{-1}N - c_N)\,x_N$$

$$\begin{bmatrix} x_0 \\ x_B \end{bmatrix} = \begin{bmatrix} c_B\,B^{-1}b \\ B^{-1}b \end{bmatrix} - \begin{bmatrix} c_B B^{-1}N - c_N \\ B^{-1}N \end{bmatrix} x_N$$

How to choose a basic admissible solutions??

Notations:

$$y_0 \equiv \begin{bmatrix} c_B\,B^{-1}b \\ B^{-1}b \end{bmatrix} \equiv \begin{bmatrix} y_{00} \\ y_{10} \\ \vdots \\ y_{m0} \end{bmatrix} \quad \text{and} \quad y_j \equiv \begin{bmatrix} c_B\,B^{-1}a_j - c_j \\ B^{-1}a_j \end{bmatrix} \equiv \begin{bmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{mj} \end{bmatrix}$$

where $a_j,\ j \in R_N$ denote the columns of matrix N.

The objective function x_0

$$x_0 = y_{00} - \sum_{j \in R_N} y_{0j}\,x_j$$

m constraints:

$$x_{Bi} = y_{i0} - \sum_{j \in R_N} y_{ij}\,x_j, \quad for\ i = 1,\ldots,m$$

First simplex table

		$-x_N$
x_0	$c_B B^{-1}b$	$c_B B^{-1}N - c_N$
x_B	$B^{-1}b$	$B^{-1}N$

$\Longrightarrow$ $c_B = 0$

		$-x_N$
x_0	0	$-c_N$
x_B	b	N

In the case, when $c_B=0$ then the top point x in the form:

$$x_N = [0,\ldots,0]^T.$$

determine the first basic admissible solution .

Linear programming problem with constraints in the form:

$$Ax \leq b,\ x \geq 0$$

1.The set X of admissible solutions $X \neq 0$

2. Algorithm simplex starts from a basic admissible solution and there is no degeneration.

I. The linear programming problem can have one solution.

$$\max_{x \in X}\ x_0 = 2\mathbf{x}_1 + 1x_2 \qquad X = \left\{ x:\ \begin{matrix} x_1 + x_2 \leq 5 \\ -x_1 + x_2 \leq 0 \\ 6x_1 + 2x_2 \leq 21 \end{matrix}\ , x \geq 0 \right\}$$

Initial simplex table

		x_1	x_2
x_0	0	-2	-1
x_3	5	1	1
x_4	0	-1	1
x_5	21	6	2

$$\hat{x} = [x_B, x_N]$$

Initial basic solution: $\hat{x} = [x_3, x_4, x_5, x_1, x_2]^T = [5, 0, 21, 0, 0]^T$

Initial value of the objective function: $x_0 = 2x_1 + x_2 \qquad x_0 = 0.$

How to improve a basic, admissible solution?

$$x_0 = y_{00} - \sum_{j \in R_N} y_{0j} x_j$$

always $x_j \geq 0,$

then when $x_k \uparrow$ *in this case* $x_0 \uparrow$ *where* $y_{0k} \leq 0$

$$x_{Bi} = y_{i0} - \sum_{j \in R_N} y_{ij} x_j, \quad for\ \ i = 1,\ldots,m$$

First simplex matrix – first basic, admissible solution.

		$\ldots$	$-x_j$	$\ldots$	$-x_k$	$\ldots$
x_0	y_{00}	$\ldots$	y_{0j}	$\ldots$	y_{0k}	$\ldots$
		$\ldots$		$\ldots$		$\ldots$
x_{Bi}	y_{iO}	$\ldots$	y_{ij}	$\ldots$	y_{ik}	$\ldots$
		$\ldots$		$\ldots$		$\ldots$
x_{Br}	y_{r0}	$\ldots$	y_{rj}	$\ldots$	y_{rk}	$\ldots$
		$\ldots$		$\ldots$		$\ldots$
x_{Bm}	y_{m0}	$\ldots$	y_{mj}	$\ldots$	y_{mk}	$\ldots$