Optimization theory and advanced computing methods

Proper local minimum

Function f(x) has a proper local minimum in point $\hat{x}$, if there exists $\delta > 0$ **such, that for** $\forall x \in E$

$$f(\hat{x}) < f(x)$$

where: $E = X \cap \Delta$

$$\Delta = \{x : 0 < \|\hat{x} - x\| < \delta\}$$

Week local minimum

Function f(x) has a week local minimum in point $\hat{x}$, if there exists $\delta_l > 0$

such that, for $\forall x \in E_l$

$$f(\hat{x}) \leq f(\hat{x}_l)$$

where:

$$E_l = X \cap \Delta_l$$

$$\Delta_l = \{x : 0 < \|\hat{x} - x\| < \delta_l\}$$

Local minimum and global minimum of a function *f(x)*

The point $\hat{x}$ determines a local minimum of a function *f(x)* in *Rⁿ space,*
When: there exists $E \subset R^n$ *for point* $\hat{x}$*, that*

$$\forall x \in E \quad f(\hat{x}) \leq f(x)$$

When $f(\hat{x}) < f(x)$ for $x \neq \hat{x}$ then the proper local minimum exists.

The point $\hat{x}$ determines a global minimum of a function *f(x)* in *R^n space,*

when $\forall x \in R^n \quad f(\hat{x}) \leq f(x)$

When $f(\hat{x}) < f(x)$ for $x \neq \hat{x}$ then the proper global minimum exists.

The Weierstrass extreme value theorem

A continuous function f(x) defined on a compact set of admissible solutions (on a closed and bounded set)

is bounded on this set and obtains its bounds.: two points exist, $x_1, x_2 \in X$

such that for each $x \in X$

the following relation is valued: $f(x_1) \leq f(x) \leq f(x_2)$

Definition of a convex set

In Euclidean space the set X $X \subset R^n$ **is convex, if for every pair of points** $x^1, x^2 \in X$ **within the set, every point on the straight line segment that joins the pairs of points is also within this set X :**

$$X = \{x : x = \lambda x^1 + (1-\lambda)x^2, 0 \leq \lambda \leq 1\}$$

Definition of a convex function

In Euclidean space the set X ($X \subset R^n$ **) is convex. Function f(x)** $f : X \to R^1$ **will be a convex function, if for every pair of points** $x^1, x^2 \in X$

and each $\lambda \in [0,1]$ **the following inequality is fulfilled:**

$$f(\lambda x^1 + (1-\lambda)x^2) \leq \lambda f(x^1) + (1-\lambda) f(x^2)$$

Definition of a strictly convex function

In Euclidean space the set X $X \subset R^n$ **is convex. Function f(x):** $f : X \to R^1$
Will be strictly convex, if for every pair of points $x^1, x^2 \in X$
And each $\lambda \in [0,1]$ **the following inequality is fulfilled:**

$$f(\lambda x^1 + (1-\lambda)x^2) < \lambda f(x^1) + (1-\lambda) f(x^2)$$

Definition of a concave function

In Euclidean space the set X ($X \subset R^n$ **) is convex. Function f(x)** $f : X \to R^1$ **will be a concave function, if for every pair of points** $x^1, x^2 \in X$

and each $\lambda \in [0,1]$ **the following inequality is fulfilled:**

$$f(\lambda x^1 + (1-\lambda)x^2) \geq \lambda f(x^1) + (1-\lambda) f(x^2)$$

Collorary: Function f(x) is concave if and only if function -f(x) is convex.

Is a function f(x) – a convex function

A matrix A (dim nxn) is called hessian, when its elements are partial derivative of second order of a function f(x):

$$A(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right]$$

Lemma 1

Let $f : X \to R^1$ **has continuous partial derivative with second order and let** $X \subset R^n$ **is a convex set. f(x) is a convex function if and only if her hessian A(x) is positive half-definite for each** $x \in X$ **.**

Definition of a positive half-definite matrix A

A matrix A is a positive half-definite matrix, when for each $x \in R^n$

$$\langle x, Ax \rangle \geq 0$$

Definition of a positive definite matrix A

A matrix A is a positive definite matrix, when for each $x \in R^n$

$$\langle x, Ax \rangle > 0$$

Sylwester Criterion – Practical testing of a function convexity

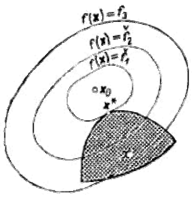
A square symmetric matrix A is positive half-definite matrix, if and only if:

$$a_{11} \geq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \geq 0 \quad ,\dots, \quad \begin{vmatrix} a_{11} & a_{12},\dots, & a_{1n} \\ a_{21} & a_{22},\dots, & a_{2n} \\ . & \dots & . \\ a_{n1} & \dots & ,a_{nn} \end{vmatrix} \geq 0$$

A square symmetric matrix A is a positive definite matrix, if and only if:

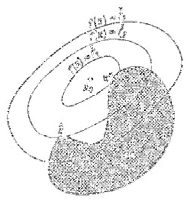
$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 \quad ,\dots, \quad \begin{vmatrix} a_{11} & a_{12},\dots, & a_{1n} \\ a_{21} & a_{22},\dots, & a_{2n} \\ . & \dots & . \\ a_{n1} & \dots & ,a_{nn} \end{vmatrix} > 0$$

Fig.1 The convex set X



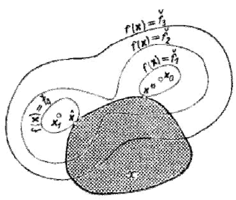
Convex function *f(x)*

Fig.2 The non-convex set of X



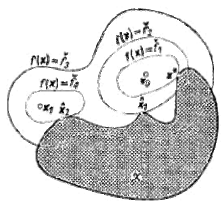
Function f(x) – a convex function.

Fig.3 The convex set of *X*



Function *f(x)* – non-convex function

Fig.4 The non-convex set of X



Function f(x) – non-convex function.

Optimization theory and advanced computing methods

Convex function

Theorem 2

Let our set X $X \subset R^n$ will be convex set and let function f(x) wil be differentiable.

Then function f(x) will be convex $\forall x, x^0 \in X$, if and only if

$$f(x) \geq f(x^0) + < \nabla f(x^0),\ x - x^0 >$$

Function f(x) is a convex function on a range (a,b) if and only if the function diagram lays above the diagram of a tangent to a curve for each point x^0 from a range (a,b).

Theorem 3

Let a function $f: X \to R^1$, for $X \subset R^n$ will be convex function, than for each real value α the set

$$X_\alpha = \{x : f(x) \leq \alpha\}$$

is convex.

Theorem 4

Let $X \subset R^n$ is a convex set . When $f_i : X \to R^1$ for i=1,...,k are convex functions and if scalar value $\alpha_i \geq 0$

for i=1,...,k then function f(x)

$$f(x) = \sum_{i=1}^{k} \alpha_i f_i(x)$$

Is a convex function.

Theorem 5

The local minimum of a convex function on a convex set $X \subset R^n$ of admissible solutions, is global minimum of that function on the set X.

Proof:

Let the function f(x) has his local minimum in point $x^1 \in X$.

It means that there exists such $\varepsilon > 0$, that:

$$f(x^1) = \min_{x \in V} f(x)$$

where: $V = \{x \in X, \|x - x^1\| \leq \varepsilon\}$

Assume, that $x^2 \in X$, $x^1 \neq x^2$. Let us consider $0 < \lambda < 1$ and

$$\lambda x^1 + (1-\lambda)x^2 \in V$$

In view of convexity of a function f(x) :

$$\lambda f(x^1) + (1-\lambda) f(x^2) \geq f(\lambda x^1 + (1-\lambda)x^2)$$

$$f(x^2) \geq \frac{f(\lambda x^1 + (1-\lambda)x^2) - \lambda f(x^1)}{1-\lambda} \geq \frac{f(x^1) - \lambda f(x^1)}{1-\lambda} = f(x^1)$$

which completes the proof.

Theorem 6

Strictly convex function f(x) defined on a convex set of admissible solutions has at most one minimum on this set.

Example

Let $c \in R^n$, $d \in R^1$. Try to show, that function *f(x)* $f(x): R^n \to R^1$ defined as follows:

$$f(x) = c^T x + d$$

Is simultaneously convex and concave function on a set of R^n.

Theorem 7

When

$$f(x): R^n \to R$$

Is concave function and X is a bounded, closed and convex set, there exists extreme point of a set X, where the function f(x) achieves the minimum value on this set.

Colorary

The linear function f(x) defined on a polyhedron (the convex set of admissible solutions) achieves it's bounds on extreme points of a polyhedron.